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ON THE COHOMOLOGY GROUPS OF A POLARISATION AND DIAGONAL QUANTISATION(¹)

BY

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ABSTRACT. The sheaf $\mathcal{S}_F(L)$ of germs of sections of a line bundle L on a manifold X covariant constant with respect to a flat connection defined for vectors in a complex subbundle F of the tangent bundle has a resolution by differential forms defined on F with values in L provided F satisfies the integrability conditions of the complex Frobenius theorem. This includes as special cases the de Rham and Dolbeault resolutions.

If there is a free S^1 action on X whose generator is tangent to F, let Y be the subset of X where parallel transport in L around the S^1 orbits is trivial. It is shown that the cohomology groups of $\mathcal{S}_F(L)$ depend only on the restriction of $\mathcal{S}_F(L)$ to Y. This is used to obtain a spectrum for a periodic Hamiltonian flow with generator in a polarisation. In the case of a classical harmonic oscillator this spectrum is found to be the same as that of the quantum mechanical oscillator.

1. **Introduction.** In [6], [8], and [22] B. Kostant and J.-M. Souriau independently developed a theory of geometric quantisation. One seeks to associate differential operators to functions on a symplectic manifold so as to preserve as much as possible of the Poisson bracket structure of the functions. Such a theory was also considered in [11]–[13] by E. Onofri and M. Pauri. For further details see also [1], [14], [15], [18], [19].

The following structure is required: If X is the manifold with symplectic form ω, one requires a line bundle L^ω over X having a connection ∇^ω whose curvature is $2\pi i\omega$ and with a parallel transport invariant Hermitian structure (often ω is said to be the curvature of ∇^ω, whilst R. Blattner [1] introduces also a constant h into these definitions). A polarisation of (X, ω) is a maximally isotropic involutive complex subtangent bundle F. If $N_F^{1/2}$ denotes the bundle

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of $\frac{1}{2}$-forms normal to F and $L = L^{\omega} \otimes N_F^{1/2}$, L has an F-connection (see §4), and one denotes by $\mathbb{S}_F$ the sheaf of germs of sections of L covariant constant along F, $S_F = \Gamma(\mathbb{S}_F)$ is the space of polarised sections.

If φ is a smooth function on X, ξ_φ denotes the Hamiltonian vector field associated to φ. An operator $\delta(\varphi)$ is defined on $\Gamma(L^{\omega})$ by $\delta(\varphi) = \nabla^{\omega}_{\xi_\varphi} + 2\pi i\varphi$ and satisfies $\delta([\varphi, \psi]) = \delta(\varphi)\delta(\psi) - \delta(\psi)\delta(\varphi)$ where the Poisson bracket $[\varphi, \psi]$ equals $\xi_\varphi(\psi)$. Let C_F^0 be all functions φ with ξ_φ a section of F and C_F^1 all functions whose Hamiltonian vector fields are infinitesimal automorphisms of F; then C_F^1 is a Lie algebra under Poisson bracket, C_F^0 is an abelian ideal in C_F^1, and if φ is in C_F^1 there is a natural Lie derivative action of ξ_φ in $\Gamma(N_F^{1/2})$. Combining $\delta(\varphi)$ with this Lie derivative gives a differential operator $\delta_F(\varphi)$ on L which preserves S_F. This action of $\delta_F(\varphi)$ on S_F is known as quantisation and is defined for functions φ in C_F^1. Functions in C_F^0 quantise as zeroth order differential operators, that is as multiplication operators, and so can be considered to have been quantised in an already diagonal form. Since one is interested in the spectrum of the quantised operators it is clearly useful when one can choose F so that φ is in C_F^0.

In [7], [8] Kostant extends the above process to functions outside C_F^1, at least when there is a second polarisation transverse to F and F is real.

D. Simms [16], [17] carries out the above construction for the isotropic harmonic oscillator in n dimensions using a polarisation F for which the Hamiltonian h is in C_F^0. However, S_F in this case vanishes, there being no smooth solutions of the polarisation equations. Simms circumvents this difficulty by considering weak solutions. Such solutions form a space on which h operates diagonally with spectrum given by the corrected Bohr-Sommerfeld condition [5] (this is due to the inclusion of $N_F^{1/2}$ in L) and with the same multiplicities as the usual quantum mechanical harmonic oscillator.

J. Sniatycki [18], [19] considers such weak polarised sections for general systems, showing that the support of the generalised sections is contained in the Bohr-Sommerfeld subset (that is the set of points x where $h(x)$ is an energy satisfying the Bohr-Sommerfeld condition). Kostant has suggested an alternative approach which avoids using weak polarised sections. Namely, if one regards S_F as $H^0(X; \mathbb{S}_F)$, then when this space vanishes one should examine the higher cohomology groups $H^p(X; \mathbb{S}_F)$. Since $\delta_F(\varphi)$ for φ in C_F^1 acts on $\mathbb{S}_F$, it also acts on each $H^p(X; \mathbb{S}_F)$, and we shall also regard this as quantisation.

It was quickly verified by Blattner, Simms, Sniatycki and the author that one certainly obtained the correct spectrum for the harmonic oscillator in one dimension on $H^1(X; \mathbb{S}_F)$. Sniatycki has considered these cohomology groups for real polarisations in [20] and [21]. In the present paper we shall treat the case of a general complex polarisation F for which C_F^0 contains a function h whose Hamiltonian flow is periodic.

In §2 we define the notion of prequantisation [6]. §§3, 4 and 5 develop a calculus of differential forms defined on F with values in L, that is, sections of $\wedge^p F^* \otimes L$, F^* being the dual bundle of F. These forms give a resolution of $\mathbb{S}_F$, due to Kostant, which gives a convenient realisation of $H^p(X; \mathbb{S}_F)$. If h in C_F^0 generates a periodic one-parameter Hamiltonian flow σ_t, we let l be the function obtained from parallel transport in L around the (closed) orbits of σ_t. In §6 we define a map J: $\Gamma(\wedge^p F^* \otimes L) \to \Gamma(\wedge^{p-1} F^* \otimes L)$, such that if ∂^F is the exterior derivative of these forms,

$$J \circ \partial^F + \partial^F \circ J = 1 - l,$$

the right-hand side meaning the operation of multiplication by the function $1 - l$. We define the Bohr-Sommerfeld subset of X to be all points x with $l(x) = 1$. If this subset is denoted by Y, we show J defines a map $\hat{J}$: $H^p(X; \mathbb{S}_F) \to H^{p-1}(Y; \mathbb{S}_{F|Y})$, and under this map $(2\pi i)^{-1}\delta_F(h)$ for h in C_F^0, which is multiplication by h on $H^p(X; \mathbb{S}_F)$, carries over to a diagonal operator on $H^{p-1}(Y; \mathbb{S}_{F|Y})$. When the set Z of orbits of σ_t on Y is a manifold, we show in §7 that we can further compose $\hat{J}$ with a map τ^{*-1} into $H^{p-1}(Z; \mathbb{S}_{\tilde{F}})$, where $\tilde{F}$ is the quotient polarisation of F and there is a line bundle $\tilde{L}$ over Z induced by L. The composite map $H^p(X; \mathbb{S}_F) \to H^{p-1}(Z; \mathbb{S}_{\tilde{F}})$ is shown to be an isomorphism for $p = 1$ and injective for $p > 1$. Finally, §8 carries through these constructions for the isotropic harmonic oscillator in n dimensions ($n > 1$) using Simms' polarisation. We show all the cohomology groups vanish except $H^1(X; \mathbb{S}_F) \cong H^0(Z; \mathbb{S}_{\tilde{F}})$. Y is, in this instance, a countable union of hypersurfaces $Y_m = h^{-1}(m + n/2)$ with m an integer. Z_m, the quotient of Y_m by σ_t, is isomorphic to $P_{n-1}(\mathbf{C})$, and $\tilde{L}|Z_m$ is the mth power of the positive generator of $H^2(P_{n-1}(\mathbf{C}); \mathbf{Z})$ so the space $H^0(Z_m; \mathbb{S}_{\tilde{F}})$ is a $\binom{n+m-1}{m}$-dimensional complex vector space. Thus, on $H^1(X; \mathbb{S}_F)$, h operates with spectrum $m + n/2$, with multiplicity $\binom{n+m-1}{m}$, with m a nonnegative integer. This agrees with Simms' results.

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2. **Prequantisation.** Let (X, ω) be a connected symplectic manifold. That is, X is a smooth, connected, finite-dimensional manifold and ω a closed 2-form which is nonsingular as a bilinear form on each tangent space. Denoting by $\mathfrak{A}(X)$ the complex vector fields on X and $\Omega^p(X)$ the complex p-forms, ω determines a linear isomorphism between $\Omega^1(X)$ and $\mathfrak{A}(X)$. The smooth functions $C(X)$ on X coincide with $\Omega^0(X)$. As usual, denote by d the exterior

derivative $\Omega^p(X) \to \Omega^{p+1}(X)$, then to each function $\varphi \in C(X)$ is associated a 1-form $d\varphi \in \Omega^1(X)$ and, thus, a vector field $\xi_\varphi \in \mathfrak{U}(X)$ where

$$d\varphi = \xi_\varphi \lrcorner \omega.$$

ξ_φ is called the Hamiltonian vector field associated to φ.

Given two functions $\varphi, \psi \in C(X)$ their Poisson bracket is defined by

$$[\varphi, \psi] = \xi_\varphi \psi = \omega(\xi_\psi, \xi_\varphi).$$

This bracket makes $C(X)$ a Lie algebra and the map $j\colon C(X) \to \mathfrak{U}(X)$ defined by $j(\varphi) = \xi_\varphi$ is a homomorphism of Lie algebras whose kernel consists of the constant functions on X.

Quantisation is the construction of representations of $C(X)$ with this Lie algebra structure, or of Lie subalgebras. The first part of the geometric quantisation procedure uses a Hermitian line bundle with connection L^ω over X, the curvature of the connection being $2\pi i\omega$, that is satisfying

$$[\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s = 2\pi i\omega(\xi, \eta)s \tag{1}$$

for any section s of L^ω and any pair ξ, η of vector fields on X. This is only possible when ω is integral, which means ω has integral periods over integral homology cycles in X. In the case when this condition is satisfied, the set of isomorphism classes of such bundles can be identified with $H^1(X, T)$ (T is the group of complex numbers of modulus one) and the prequantisation of $C(X)$ is constructed on the space $\Gamma(L^\omega)$ of smooth sections of L^ω as follows. To each function $\varphi \in C(X)$ we associate a first order operator $\delta(\varphi)\colon \Gamma(L^\omega) \to \Gamma(L^\omega)$ by setting

$$\delta(\varphi) = \nabla_{\xi_\varphi} + 2\pi i\varphi.$$

By virtue of equation (1) δ satisfies

$$\delta([\varphi, \psi]) = \delta(\varphi)\delta(\psi) - \delta(\psi)\delta(\varphi)$$

for $\varphi, \psi \in C(X)$, so δ is a homomorphism of Lie algebras, where the operators on L^ω are given their usual commutator bracket Lie algebra structure. Further details may be found in [1], [6], [8], [14], [15].

3. **The Poincaré lemma.** A subtangent bundle on a smooth manifold X is a smooth subbundle F of the complexified tangent bundle TX^c. Let $\bar{F}$ be the conjugate of F with respect to the real form TX of TX^c which is also a subtangent bundle. The codimension of F will be $\dim_{\mathbf{C}} T_x X^c - \dim_{\mathbf{C}} F_x$.

Denote by $\mathfrak{U}_F(U)$ the space of smooth sections of F over the open set $U \subset X$. F is said to be involutive if $\mathfrak{U}_F(X)$ is a Lie subalgebra of $\mathfrak{U}(X)$

$= \mathcal{U}_{TX^c}(X)$. For any open set $U \subset X$ let $C_F(U)$ denote all the functions φ on U with $\xi(\varphi) = 0$ for all ξ in $\mathcal{U}_F(U)$. The spaces $C_F(U)$ together with restriction maps form a presheaf; let $\mathcal{C}_F$ denote the associated sheaf.

Let $F^0 \subset T^*X^c$ be the subcotangent bundle of covectors which vanish on F. Thus $\varphi \in C_F(U)$ if and only if $d\varphi$ is a section of F^0 on U. If F has codimension m then $\wedge^m F^0 = K^F$ is a complex line bundle over X which we call the canonical bundle of F.

If there are functions $\varphi_1, \ldots, \varphi_m$ in $C_F(U)$ with $(d\varphi_1, \ldots, d\varphi_m)$ a frame of F^0 at each point of U, then $(\varphi_1, \ldots, \varphi_m)$ is called a C_F-coordinate system and U a C_F-coordinate neighbourhood. F is said to be integrable if X can be covered by C_F-coordinate neighbourhoods. Note that if F is integrable, it is automatically involutive. The converse is not, in general, true.

THEOREM 1 (FROBENIUS AND NIRENBERG). *If F is involutive and if either*
(i) $\dim_{\mathbf{C}} F_x \cap \bar{F}_x$ *is constant and* $F + \bar{F}$ *is involutive*; *or*
(ii) *X is a real analytic manifold and F an analytic subtangent bundle,*
then F is integrable.

PROOF. (i) This case is just the complex Frobenius theorem proved by Nirenberg [10] and Hörmander [4].

(ii) We reduce this to case (i) as follows. The result is local so we only need prove that a specific point x in X has a C_F-coordinate neighbourhood. Without loss of generality we may suppose X is $\mathbf{R}^{n+m}$, x is the origin and F is given by n analytic vector fields $\xi_1, \ldots, \xi_n$ which can be expressed in terms of coordinates $x_1, \ldots, x_{n+m}$ by

$$\xi_j = \sum_{k=1}^{n+m} a_{kj}(x_1, \ldots, x_{n+m}) \frac{\partial}{\partial x_k}, \qquad j = 1, \ldots, n.$$

Each a_{kj} is an analytic function of $x_1, \ldots, x_{n+m}$, and so extends by analytic continuation to a holomorphic function $\hat{a}_{kj}$ on $\mathbf{C}^{m+n}$. That F be involutive means there are analytic functions c_{jkl} such that

$$[\xi_j, \xi_k] = \sum_{l=1}^{n} c_{jkl} \xi_l, \qquad j, k = 1, \ldots, n.$$

Let $\hat{c}_{jkl}$ be the analytic continuations to $\mathbf{C}^{m+n}$ of the c_{jkl} and define holomorphic vector fields $\hat{\xi}_j$ on $\mathbf{C}^{m+n}$ by

$$\hat{\xi}_j = \sum_{k=1}^{m+n} \hat{a}_{kj} \frac{\partial}{\partial z_k}, \qquad j = 1, \ldots, n;$$

then it follows that

$$[\hat{\xi}_j, \hat{\xi}_k] = \sum_{l=1}^{n} \hat{c}_{jkl} \hat{\xi}_l, \qquad j, k = 1, \ldots, n,$$

and hence the distribution $\hat{F}$ on $\mathbf{C}^{m+n}$ spanned by $\hat{\xi}_1, \ldots, \hat{\xi}_n$ and $\partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_{m+n}$ is smooth, involutive and of codimension m. Moreover $\hat{F}_z \cap \bar{\hat{F}}_z$ has constant dimension $2n$ and $\hat{F} + \bar{\hat{F}}_z$ is involutive since it equals all of $(T\mathbf{C}^{m+n})^c$. Thus part (i) implies the existence of a $C_{\hat{F}}$-coordinate system $\psi_1, \ldots, \psi_m$ on a neighbourhood of the origin in $\mathbf{C}^{m+n}$. The restrictions $\varphi_1, \ldots, \varphi_m$ to $\mathbf{R}^{m+n}$ give the required C_F-coordinate system on a neighbourhood of the origin.

The validity of the theorem in case (ii) may be useful in the case X is a homogeneous space for a Lie group G and F is G-invariant, since in this case X and F are then analytic.

Denote by $\Omega_F^p(U)$ the sections of $\wedge^p F^*$ over an open set U in X. $\Omega_F^0(U)$ is just $C(U)$ the space of smooth functions on U. If F is involutive we can define a differential $d^F\colon \Omega_F^p(U) \to \Omega_F^{p+1}(U)$ in the usual way, namely regarding sections of $\wedge^p F^*$ as alternating $C(U)$-multilinear maps of $\mathfrak{U}_F(U)$ into $C(U)$, we define, for $\alpha \in \Omega_F^p(U)$,

$$(d^F\alpha)(\xi_1, \ldots, \xi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \xi_i[\alpha(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1})] + \sum_{i<j} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})$$

where $\xi_i \in \mathfrak{U}_F(U)$ for $i = 1, \ldots, p+1$. Then $(d^F)^2 = 0$ and $d^F\varphi = d\varphi|F$ for $\varphi \in C(X)$. In fact $C_F(U) = \{\varphi \in C(U) | d^F\varphi = 0\}$. The differential complex

$$\Omega_F^0(X) \xrightarrow{d^F} \Omega_F^1(X) \xrightarrow{d^F} \cdots \xrightarrow{d^F} \Omega_F^n(X) \xrightarrow{d^F} 0$$

has cohomology groups which we denote by $H^p(\Omega_F^*(X))$.

The sheaf $\mathcal{C}_F$ also has cohomology groups $H^p(X; \mathcal{C}_F)$ associated to it. Notice that

$$H^0(X; \mathcal{C}_F) = \Gamma(\mathcal{C}_F) = C_F(X) = \operatorname{Ker} d^F\colon C(X) \to \Omega_F^1(X) = H^0(\Omega_F^*(X)).$$

In fact we also have

THEOREM 2. *If condition* (i) *of Theorem* 1 *holds for an involutive F then* $H^p(X; \mathcal{C}_F)$ *is isomorphic to* $H^p(\Omega_F^*(X))$ *for all p. More is true: if* $\mathcal{Q}_F^p$ *denotes the sheaf of germs of local smooth sections of* $\wedge^p F^*$ *with induced maps* $d^F\colon \mathcal{Q}_F^p \to \mathcal{Q}_F^{p+1}$, *and noting that* $\mathcal{C}_F = \operatorname{Ker} d^F\colon \mathcal{Q}_F^0 \to \mathcal{Q}_F^1$, *we have*

$$0 \to \mathcal{C}_F \hookrightarrow \mathcal{Q}_F^0 \xrightarrow{d^F} \mathcal{Q}_F^1 \xrightarrow{d^F} \cdots \xrightarrow{d^F} \mathcal{Q}_F^n \xrightarrow{d^F} 0 \tag{2}$$

is a fine resolution of $\mathcal{C}_F$ *giving rise to the above isomorphisms.*

PROOF. The sheaves $\mathcal{Q}_F^p$ are clearly fine, and since $(d^F)^2 = 0$, it is only necessary to prove a Poincaré lemma for d^F. That is, we must show, given an open set U in X and a section α of $\wedge^p F^*$, $p \geqslant 1$, on U with $d^F\alpha = 0$, then any point x in U has a neighbourhood V on which there is a section β of $\wedge^{p-1}F^*$ and $d^F\beta = \alpha|V$. Without loss of generality we may suppose U is a C_F-coordinate neighbourhood with C_F-coordinates $v_1, \ldots, v_l, z_{l+1}, \ldots, z_m$, where $z_j = x_j + iy_j$ and $dv_1, \ldots, dv_l, dx_{l+1}, \ldots, dx_m, dy_{l+1}, \ldots, dy_m$ are linearly independent at each point of U. We may add $k = n + l - m$ real functions $u_1, \ldots, u_k$ to obtain a coordinate system $u_1, \ldots, u_k, v_1, \ldots, v_l, x_{l+1}, \ldots, x_m, y_{l+1}, \ldots, y_m$ on U. F is then spanned on U by $\partial/\partial u_1, \ldots, \partial/\partial u_k, \partial/\partial \bar{z}_{l+1}, \ldots, \partial/\partial \bar{z}_m$. If $k = 0$ Chern's proof [2] of the Dolbeault lemma adapts to this situation, whilst if $k = n$ the proof of the real Poincaré lemma [23] will work. For $0 < k < n$ it is necessary to combine the two proofs as follows:

Let $\varphi_i = v_i$, $i = 1, \ldots, l$, and $\varphi_i = z_i$, $i = l + 1, \ldots, m$, be the C_F-coordinates described above which will remain fixed throughout the argument. Also let $\psi_i = u_i$, $i = 1, \ldots, k$, and $\psi_i = \bar{z}_{i+l-k}$, $i = k + 1, \ldots, n$; then $d^F\psi_1, \ldots, d^F\psi_n$ give a basis for F^* on U. We shall say an element $\alpha \in \Omega_F^p(U)$ is j-dependent if the expansion $\alpha = \sum_{i_1 \cdots i_p} a_{i_1, \ldots, i_p} d^F\psi_{i_1} \wedge \cdots \wedge d^F\psi_{i_p}$ involves only $d^F\psi_i$ with $i \leqslant j$. We make the following hypothesis:

(H_j) If α is j-dependent on U with $d^F\alpha = 0$ and x in U there is a neighbourhood V of x in U and a form β on V which is j-dependent with $d^F\beta = \alpha|V$.

We are trying to prove (H_n) whilst for nonzero α the least value which j can take is the degree p of α. In this case $\alpha = f d^F\psi_1 \wedge \cdots \wedge d^F\psi_p$ and $d^F\alpha = 0$ if and only if $\xi_j(f) = 0$ for all $j > p$ where $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m$ is the dual frame field to $d\psi_1, \ldots, d\psi_n, d\varphi_1, \ldots, d\varphi_m$. If $j \leqslant k$, ξ_j is $\partial/\partial u_j$ and if $j > k$, ξ_j is $\partial/\partial \bar{z}_{j+l-k}$. In either case it is possible to find a function g on a neighbourhood V of x with $\xi_p(g) = f$ and $\xi_j(g) = 0, j > p$. Then $\alpha = \xi_p(g) d^F\psi_1 \wedge \cdots \wedge d^F\psi_p = d^F[(-1)^{p-1} g d^F\psi_1 \wedge \cdots \wedge d^F\psi_{p-1}]$ on V which proves (H_p).

To prove (H_n) we proceed by induction on j. That is, we assume (H_j) proven and suppose α is a $j + 1$-dependent p-form on U with $d^F\alpha = 0$ and x is in U. We can write $\alpha = \alpha_1 + d^F\psi_{j+1} \wedge \alpha_2$ with α_1 and α_2 being j-dependent forms on U. It follows from $d^F\alpha = 0$ that $\xi_k(\alpha_2) = 0$ for $k > j + 1$, where by $\xi_k(\alpha_2)$ we mean the form $\sum_{i_1 \cdots i_{p-1}} \xi_k(a_{i_1, \ldots, i_{p-1}}) d^F\psi_{i_1} \wedge \cdots \wedge d^F\psi_{i_{p-1}}$ if $\alpha_2 = \sum_{i_1 \cdots i_{p-1}} a_{i_1 \cdots i_{p-1}} d^F\psi_{i_1} \wedge \cdots \wedge d^F\psi_{i_{p-1}}$. As above we can find a j-dependent $p - 1$-form β_1 on a neighbourhood V_1 of x in U with $\alpha_2 = \xi_{j+1}(\beta_1)$ and $\xi_k(\beta_1) = 0$, $k > j + 1$. But then

$$\begin{aligned} d^F\beta_1 &= \sum_{i=1}^{n} d^F\psi_i \wedge \xi_i(\beta_1) = \sum_{i=1}^{j} d^F\psi_i \wedge \xi_i(\beta_1) + d^F\psi_{j+1} \wedge \alpha_2 \\ &= \beta_2 + d^F\psi_{j+1} \wedge \alpha_2 \end{aligned}$$

with β_2 a j-dependent p-form on V_1. Thus $\alpha = \alpha_1 + d^F\beta_1 - \beta_2$ and $\alpha_1 - \beta_2$ is j-dependent. From $d^F\alpha = 0$ follows $d^F(\alpha_1 - \beta_2) = 0$, and since we are assuming (H_j), it follows that $\alpha_1 - \beta_2 = d^F\beta_3$ on some neighbourhood V of x in V_1, and then if $\beta = \beta_1 + \beta_3$, we have $d^F\beta = \alpha|V$. Thus (H_j) implies (H_{j+1}) and, hence, the desired result.

We use Theorem 2 to identify $H^p(X; \mathcal{C}_F)$ and $H^p(\Omega_F^*(X))$.

COROLLARY 1. *Assume the same conditions as in Theorem* 2. *Let* $\omega \in \Omega^2(X)$ *be a given closed* 2*-form whose restriction to F vanishes. Then each x in X has a neighbourhood U with a* 1*-form* α_0 *such that* (i) $\alpha_0|F = 0$ *and* (ii) $d\alpha_0 = \omega|U$.

PROOF. Since $d\omega = 0$, x has a neighbourhood U_1 on which there is a 1-form α_1 with $d\alpha_1 = \omega|U_1$. Let $\tilde{\alpha}_1$ be the restriction of α_1 to F. Then $d^F\tilde{\alpha}_1 = 0$. Thus there is a neighbourhood U of x in U_1 and a function φ on U with $d^F\varphi = \tilde{\alpha}_1|U$. Let $\alpha_0 = \alpha_1|U - d\varphi$. Then $d\alpha_0 = \omega|U$ and $\alpha_0|F = \tilde{\alpha}_1|U - d^F\varphi = 0$.

REMARK 1. Theorem 2 does not remain true if instead we assume condition (ii) of Theorem 1. For suppose $X = \mathbf{R}^3$ and F is spanned by the vector field

$$P = \tfrac{1}{2}(\partial/\partial x_1 + i\partial/\partial x_2) - i(x_1 + ix_2)\partial/\partial x_3.$$

Clearly F is an analytic complex distribution. Lewy [9] shows there are C^∞ functions f on X with the property that $P\varphi = f$ has no C^∞ solution on any open set. Choose such a function f and let β be the section of F^* defined by $\beta(P) = f$. Then $d^F\beta = 0$, but there are no local solutions to $\beta = d^F\varphi$.

REMARK 2. If $F = TX^c$, Theorem 2 is the usual de Rham theorem since $\mathcal{C}_F$ is the constant sheaf $\underline{\mathbf{C}}$. If TX^c is the internal direct sum $F \oplus \overline{F}$, then defining $J \in \operatorname{End} TX$ as $-i$ on F and i on $\overline{F}$, X becomes a complex manifold, C_F-coordinates are holomorphic coordinate systems and $\mathcal{C}_F$ is the sheaf of germs of holomorphic functions. Theorem 2 in this case contains the $\bar{\partial}$ resolution of $\mathcal{C}_F$, $\Omega_F^p(X)$ easily being identified with $\Omega^{0,p}(X)$ the forms of type $(0,p)$.

We shall say an involutive tangent subbundle is strongly integrable if it satisfies condition (i) of Theorem 1.

EXAMPLE 1. Let (X, ω) be a symplectic manifold. A polarisation of (X, ω) is an involutive subtangent bundle F of TX^c with $\omega_x(\xi, \eta) = 0$ for all $\xi, \eta \in F_x$, $x \in X$ and $\dim_{\mathbf{C}} F_x = \frac{1}{2}\dim X = n$. F is said to be admissible if it is strongly integrable. The corollary to Theorem 2 implies the existence of a 1-form α_0 on some neighbourhood U of any point with $\omega|U = d\alpha_0$ and $\alpha_0 \in \Gamma(F^0|U)$. F is said to be real if $F = \overline{F}$. Then $F = D^c$ is the complexification of a real integrable distribution D. Let $U_1 \subset U$ be an open set on which there exist C_F-coordinates $q_1, \ldots, q_n$; then $dq_1, \ldots, dq_n$ form a frame at each point of U_1 for F^0, so there are functions $p_1, \ldots, p_n$ with $\alpha_0|U_1 = \sum_{i=1}^n p_i dq_i$. We may assume the p_i's and q_i's are real; then $\omega|U_1 = d\alpha_0|U_1 = \sum_{i=1}^n dp_i \wedge dq_i$

implies $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ form a Darboux coordinate system on U_1. Moreover $(\partial/\partial p_1, \ldots, \partial/\partial p_n)$ form a frame for D at each point of U_1.

4. **Line bundle valued forms.** Let $\pi\colon L \to X$ be a line bundle over X and F a subtangent bundle. An F-connection in L is a linear map

$$\nabla\colon \Gamma(L) \to \Gamma(F^* \otimes L)$$

such that for any section s of L and smooth function φ on X,

$$\nabla(\varphi s) = \varphi \nabla s + d^F\varphi \otimes s.$$

For $\xi \in \mathfrak{U}_F(X)$ we define $\nabla_\xi \in \operatorname{End}\Gamma(L)$ by $\nabla_\xi s = (\nabla s)(\xi)$ regarding $F^* \otimes L$ as $\operatorname{Hom}(F, L)$. When F is involutive the curvature of ∇ is the form $\rho \in \Omega_F^2(X)$ such that

$$[\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s = \rho(\xi, \eta)s$$

for all ξ, η in $\mathfrak{U}_F(X)$. We extend ∇ to a map $\partial^F\colon S_F^p(L) \to S_F^{p+1}(L)$ where $S_F^p(L) = \Gamma(\wedge^p F^* \otimes L)$ (so $\nabla\colon S_F^0(L) \to S_F^1(L)$) by defining

$$(\partial^F\alpha)(\xi_1, \ldots, \xi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{\xi_i}[\alpha(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1})]$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})$$

where $\xi_i \in \mathfrak{U}_F(X)$, $i = 1, \ldots, p+1$.

There is also an action of $\Omega_F^p(X)$ on $S_F^q(L)$ into $S_F^{p+q}(L)$, denoted by $\alpha \wedge \beta$, making $S_F^*(L) = \sum_{p\geq 0} S_F^p(L)$ into a left $\Omega_F^*(X) = \sum_{p \geq 0} \Omega_F^p(X)$-module. Then we have for any $\alpha \in S_F^*(L)$, $\beta \in \Omega_F^p(X)$,

$$\partial^F(\partial^F\alpha) = \rho \wedge \alpha; \quad \partial^F(\beta \wedge \alpha) = (d^F\beta) \wedge \alpha + (-1)^p \beta \wedge \partial^F\alpha.$$

Thus $(\partial^F)^2 = 0$ if $\rho = 0$, in which case we say ∇ is flat.

EXAMPLE 2. The canonical bundle $K = K^F = \wedge^m F^0$ of F has a flat F-connection as follows: $\Gamma(K) \subset \Omega^m(X)$ so $d\alpha$ is an $m+1$-form on X for $\alpha \in \Gamma(K)$. Let $\Omega_F^{m,p}(X)$ be all $m+p$-forms $\beta \in \Omega^{m+p}(X)$ with $\alpha \wedge \beta = 0$ for all α in $\Gamma(F^0)$. Then $\Omega_F^{m,0}(X) = \Gamma(K)$ and $d\Omega_F^{m,p}(X) \subset \Omega_F^{m,p+1}(X)$. There is a canonical isomorphism $\Omega_F^{m,p}(X) \xrightarrow{\sim} \Gamma(\wedge^p F^* \otimes K) = S_F^p(K)$ defined by $\alpha \wedge \beta \to \alpha | F \otimes \beta$ for α in $\Omega^p(X)$, β in $\Gamma(K)$. Thus $\Omega_F^{m,1}(X)$ is isomorphic to $S_F^1(K)$. Let ∇ be the composition

$$\Gamma(K) \xrightarrow{d} \Omega_F^{m,1}(X) \cong S_F^1(K) = \Gamma(F^* \otimes K).$$

This is the required F-connection, which is flat since $d^2 = 0$. The differential $\partial^F\colon S_F^p(K) \to S_F^{p+1}(K)$ coincides with

$$S_F^p(K) \cong \Omega_F^{m,p}(X) \xrightarrow{d} \Omega_F^{m,p+1}(X) \cong S_F^{p+1}(K).$$

Returning to the general situation with F an involutive tangent subbundle and ∇ a flat F-connection in a line bundle L over X, the spaces $S_{F|U}^p(L|U)$ form presheaves and maps of presheaves $\partial^F: S_{F|U}^p(L|U) \to S_{F|U}^{p+1}(L|U)$ which give rise to sheaves $\mathbb{S}_F^p(L)$ and homomorphisms $\partial^F: \mathbb{S}_F^p(L) \to \mathbb{S}_F^{p+1}(L)$. Denote by $\mathbb{S}_F$ the kernel of $\partial^F: \mathbb{S}_F^0(L) \to \mathbb{S}_F^1(L)$. $S_{F|U}^0(L|U)$ is the space of sections of L over U and, thus, $\mathbb{S}_F^0(L)$ the sheaf of germs of sections of L. $\mathbb{S}_F$ is the sheaf of germs of sections of L which are covariant constant along F. We have thus a sequence

$$(3) \qquad 0 \to \mathbb{S}_F \hookrightarrow \mathbb{S}_F^0(L) \xrightarrow{\partial^F} \mathbb{S}_F^1(L) \xrightarrow{\partial^F} \cdots \xrightarrow{\partial^F} \mathbb{S}_F^n(L) \xrightarrow{\partial^F} 0$$

with $\partial^F \circ \partial^F = 0$.

THEOREM 3. *Let F be an involutive, strongly integrable subtangent bundle on X and L a line bundle over X with a flat F-connection ∇; then the sequence* (3) *is a fine resolution of $\mathbb{S}_F$.*

PROOF. This follows from the exactness of the sequence (2) which is guaranteed by Theorem 2. It is necessary only to show that if $\alpha \in S_{F|U}^p(L|U)$ satisfies $\partial^F\alpha = 0$ and $x \in U$, then there is a neighbourhood V of x in U and $\beta \in S_{F|V}^{p-1}(L|V)$ with $\partial^F\beta = \alpha|V$. Thus suppose $\alpha \in S_{F|U}^p(L|U)$ with $\partial^F\alpha = 0$ and $x \in U$. There is a neighbourhood V_1 of x in U with a nowhere vanishing section s_0: $V_1 \to L$. Then one has $\alpha_0 \in \Omega_F^1(V_1)$ defined by $\nabla s_0 = \alpha_0 \otimes s_0$. Moreover the flatness of ∇ implies $d^F\alpha_0 = 0$. Thus by Theorem 2 there is a neighbourhood V_2 of x in V_1 and a function φ_0 on V_2 with $d^F\varphi_0 = \alpha_0|V_2$. Set $s_1 = e^{-\varphi_0}s_0|V_2$; then $\nabla s_1 = 0$ and s_1 vanishes nowhere on V_2. Thus there is α_1 in $\Omega_F^p(V_2)$ with $\alpha|V_2 = \alpha_1 \otimes s_1$, and $\partial^F\alpha|V_2 = (d^F\alpha_1) \otimes s_1$. $\partial^F\alpha = 0$ implies $d^F\alpha_1 = 0$ and so there is a neighbourhood V of x in V_2 and β_1 in $\Omega_F^{p-1}(V)$ with $\alpha_1|V = d^F\beta_1$. Putting $\beta = \beta_1 \otimes s_1|V$, $\partial^F\beta = d^F\beta_1 \otimes s_1|V = (\alpha_1 \otimes s_1)|V = \alpha|V$, which proves the result.

COROLLARY 2. *Under the same conditions as in the theorem, for any x in X there is a neighbourhood U of x and a section s_1: $U \to L$ which vanishes nowhere and $\nabla s_1 = 0$ on U.*

PROOF. The section s_1 defined in the course of the proof of the theorem with $U = V_2$ satisfies these requirements.

The cohomology groups $H^p(X; \mathbb{S}_F)$ of the sheaf $\mathbb{S}_F$ can now be identified with those of the complex

$$(4) \qquad S_F^0(L) \xrightarrow{\partial^F} S_F^n(L) \xrightarrow{\partial^F} \cdots \xrightarrow{\partial^F} S_F^n(L).$$

One consequence of this is that these groups vanish in degrees higher than the fibre dimension of F.

REMARK 3. Let $TX^c = F \oplus \bar{F}$ and give X the complex structure as in Remark 2. A line bundle $\pi: L \to X$ is holomorphic if there is a covering $\{U_i\}$ of X and nowhere vanishing sections $s_i: U_i \to L$ with $s_j = c_{ij}s_i$ on $U_i \cap U_j$ and $c_{ij} \in C_F(U_i \cap U_j)$. That is, the transition functions c_{ij} are holomorphic. Then L has a flat F-connection ∇ defined by $\nabla s = d^F f_i \otimes s_i$ on U_i if $s = f_i s_i$ on U_i. A section s of L is holomorphic if and only if $\nabla s = 0$. $\mathcal{S}_F$ is the sheaf of germs of holomorphic sections of L and (3) the $\bar{\partial}$ resolution of $\mathcal{S}_F$.

EXAMPLE 3. Returning to the situation of Example 1, let $\pi: L \to X$ be a line bundle with flat F-connection ∇ (L may be the line bundle with connection of prequantisation of §2 or such a line bundle tensored with a line bundle of $\frac{1}{2}$-forms normal to F which has a canonical flat F-connection [1], [8], [15], [18]). $\mathcal{S}_F$ is the sheaf of germs of polarised sections of L. $H^0(X; \mathcal{S}_F)$, denoted S_F, is the space of global polarised sections of L and is the module on which quantisation [8] takes place. When S_F vanishes, Kostant has suggested using the higher cohomology groups of $\mathcal{S}_F$ for the quantisation process. (4) then gives a convenient representation of these cohomology groups in terms of L-valued forms defined on F. This example will be considered further in the later sections.

5. **Lie derivatives.** Let F be a subtangent bundle on a manifold X. An infinitesimal automorphism of F is a vectorfield ξ such that $[\xi, \eta] \in \mathfrak{A}_F(X)$ for all $\eta \in \mathfrak{A}_F(X)$. Then for any $\alpha \in \Omega_F^p(X)$ we define the Lie derivative $\theta(\xi)\alpha$ by

$$(\theta(\xi)\alpha)(\xi_1, \ldots, \xi_p) = \xi[\alpha(\xi_1, \ldots, \xi_p)] - \sum_{i=1}^{p} \alpha(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_p)$$

for $\xi_1, \ldots, \xi_p$ in $\mathfrak{A}_F(X)$. Each ξ in $\mathfrak{A}_F(X)$ is an infinitesimal automorphism of F if F is involutive; then defining $i(\xi): \Omega_F^p(X) \to \Omega_F^{p-1}(X)$ by

$$(i(\xi)\alpha)(\xi_1, \ldots, \xi_{p-1}) = \alpha(\xi, \xi_1, \ldots, \xi_{p-1})$$

for $\xi_1, \ldots, \xi_{p-1}$ in $\mathfrak{A}_F(X)$ if $p \geqslant 1$ and $i(\xi)\varphi = 0$ for $\varphi \in \Omega_F^0(X)$, then

$$\theta(\xi) = d^F \circ i(\xi) + i(\xi) \circ d^F$$

for $\xi \in \mathfrak{A}_F(X)$.

If ξ generates a one-parameter group σ_t of diffeomorphisms, that is

$$\xi\varphi = (d/dt)\sigma_{-t}^* \varphi|_{t=0}, \qquad \varphi \in C(X),$$

then

(5) $$\theta(\xi)\alpha = (d/dt)\sigma^*_{-t}\alpha|_{t=0}, \qquad \alpha \in \Omega^*_F(X),$$

where $\sigma^*\alpha$ is defined for diffeomorphisms σ with $\sigma_* F_x = F_{\sigma x}$ by

$$(\sigma^*\alpha)_x(\xi_1, \ldots, \xi_p) = \alpha_{\sigma x}(\sigma_*\xi_1, \ldots, \sigma_*\xi_p)$$

for $\alpha \in \Omega^p_F(X)$, $\xi_1, \ldots, \xi_p \in F_x$.

Now let $\pi: L \to X$ be a line bundle with flat F-connection ∇ and F an involutive subtangent bundle. If $\xi \in \mathcal{U}_F(X)$ and $\alpha \in S^p_F(L)$, define $\theta(\xi)\alpha \in S^p_F(L)$ by

$$(\theta(\xi)\alpha)(\xi_1, \ldots, \xi_p) = \nabla_\xi[\alpha(\xi_1, \ldots, \xi_p)] - \sum_{i=1}^p \alpha(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_p)$$

for $\xi_1, \ldots, \xi_p$ in $\mathcal{U}_F(X)$. Then one has the identity

$$\theta(\xi) = \partial^F \circ i(\xi) + i(\xi) \circ \partial^F$$

where $i(\xi)$: $S^p_F(L) \to S^{p-1}_F(L)$ is defined in the obvious way.

Suppose $\xi \in \mathcal{U}_F(X)$ generates a one-parameter group σ_t; then to obtain an analogue of (5) we proceed as follows. Since $\xi \in \mathcal{U}_F(X)$ the curve $\gamma_x(t) = \sigma_{-t}x$ through x has tangent at each point in F so that parallel transport along γ_x in L is defined with respect to ∇. Let $P_{x,t}$ denote the transport map: $L_{\sigma_t x} \to L_x$ along γ_x from $\sigma_t x$ to x. For α in $\Omega^p_F(X)$ define $\Sigma_t\alpha$ in $\Omega^p_F(X)$ by

$$(\Sigma_t\alpha)_x(\xi_1, \ldots, \xi_p) = P_{x,t}[\alpha_{\sigma_t x}(\sigma_t x(\sigma_{t*}\xi_1, \ldots, \sigma_{t*}\xi_p))]$$

for $\xi_1, \ldots, \xi_p$ in F_x. Then we have

(6) $$\theta(\xi)\alpha = (d/dt)\Sigma_{-t}\alpha|_{t=0},$$

using for example the formula given in [6] for parallel transport in line bundles.

6. The periodic case. Continuing with the notation of §5, suppose now σ_t is a periodic one-parameter group so $\sigma_t = \sigma_{1+t}$ for all t in $\mathbf{R}$. The lifting Σ_t into $S^p_F(L)$, although it is a group, is no longer periodic in general. However Σ_1 is related to Σ_0 as follows: Since $\sigma_1 = \sigma_0 = \mathrm{id}_X$ the curve γ_x is closed. Let $l(x)$ be the scalar in $\mathrm{GL}(1, \mathbf{C})$ which equals $P_{x,1}: L_x \to L_x$. Then for α in $S^p_F(L)$,

$$(\Sigma_1\alpha)_x(\xi_1, \ldots, \xi_p) = P_{x,1}[\alpha_{\sigma_1 x}(\sigma_{1*}\xi_1, \ldots, \sigma_{1*}\xi_p)] = l(x)\alpha_x(\xi_1, \ldots, \xi_p)$$

for $\xi_i \in F_x$, $i = 1, \ldots, p$. So $\Sigma_1\alpha = l\alpha$. Thus

$$(l-1)\alpha = \Sigma_1\alpha - \Sigma_0\alpha = \int_0^1 \frac{d}{ds}\Sigma_s\alpha\,ds = \int_0^1 \frac{d}{dt}\Sigma_{s+t}\alpha|_{t=0}\,ds$$

$$= \int_0^1 \Sigma_s\frac{d}{dt}\Sigma_t\alpha|_{t=0}\,ds = \int_0^1 \Sigma_s[-\theta(\xi)\alpha]\,ds.$$

Now one can easily check that $\Sigma_s \circ \theta(\xi) = \theta(\xi) \circ \Sigma_s$ and $\partial^F \circ \Sigma_s = \Sigma_s \circ \partial^F$ so that

$$(l - 1)\alpha = -\theta(\xi) \int_0^1 \Sigma_s \alpha \, ds,$$

and defining $I\alpha$ by $\int_0^1 \Sigma_s \alpha \, ds$, we have

$$(1 - l)\alpha = \partial^F \circ i(\xi) \circ I\alpha + i(\xi) \circ I \circ \partial^F \alpha.$$

If $J = i(\xi) \circ I$,

$$J \circ \partial^F \alpha + \partial^F \circ J\alpha = (1 - l)\alpha \tag{7}$$

where the right-hand side of (7) is the operation of multiplying a form by the function $1 - l$. This equation certainly makes sense for all α in $S_F^p(L)$ if $p \geqslant 1$, whilst for $p = 0$ it becomes

$$J \circ \partial^F s = (1 - l)s, \qquad s \in \Gamma(L). \tag{8}$$

Usually the bundles with which we deal have a parallel-transport invariant Hermitian structure so that the function l takes its values in the unit circle T (see [6]). We shall suppose this is so and that 1 is a regular value of l as a map $l: X \to T$. Then $l^{-1}(1) = Y$ is a closed submanifold of X of codimension one which we shall call the Bohr-Sommerfeld subset of the flow σ_t.

PROPOSITION 1. *Under the above assumptions* $S_F = 0$.

PROOF. S_F consists of all solutions of $\partial^F s = 0$ with s in $\Gamma(L)$. But if $\partial^F s = 0$ we have $(1 - l)s = J \circ \partial^F s = 0$ and so $s = 0$ on $X - Y$. By continuity, since a codimension one manifold has dense complement, we have $s = 0$ everywhere.

THEOREM 4. *Again with the above assumptions, if* $\pi: E \to X$ *is a vector bundle, then a section (smooth)* s *of* E *vanishes on* Y *if and only if there is a smooth section* r *of* E *with* $s = (1 - l)r$.

PROOF. Clearly, if $s = (1 - l)r$, then s vanishes on Y. Suppose the converse has been proven for functions on X. We can cover X by open sets U on which there is a smooth frame $s_1, \ldots, s_m$ for E; then on U, $s = \sum_{i=1}^m f_i s_i$ for smooth functions $f_1, \ldots, f_m$. Moreover, s vanishes on $U \cap Y$ if and only if all the functions f_i do. Then there are functions g_i with $f_i = (1 - l)g_i$ and $s = (1 - l) \sum_{i=1}^m g_i s_i$. One can easily see that the local sections $\sum_{i=1}^m g_i s_i$ piece together to give a global section r satisfying our requirements. It remains to prove the result for functions on sufficiently small open sets.

Since 1 is a regular value of l there are coordinates $x_1, \ldots, x_N$ on a convex open set U in X with $Y \cap U$ given by $x_1 = 0$ and $l = e^{ix_1}$ on U. Suppose φ in $C(U)$ vanishes on $Y \cap U$. Now $\partial l/\partial x_1 = il$ is nonzero on U. The integral, defined for y in U,

$$\int_0^1 \left(\frac{\partial l}{\partial x_1}\right)(tx_1(y), x_2(y), \ldots, x_N(y))\, dt$$

becomes $\partial l/\partial x_1(y)$ for y in $Y \cap U$, which is nonzero. Thus by continuity, by shrinking U if necessary, we can suppose the integral is nonzero throughout U. Then, if φ vanishes on $U \cap Y$,

$$\frac{\int_0^1 (\partial\varphi/\partial x_1)(tx_1(y), x_2(y), \ldots, x_N(y))\, dt}{\int_0^1 (\partial l/\partial x_1)(tx_1(y), x_2(y), \ldots, x_N(y))\, dt}$$

defines a smooth function on U which we denote by $-\psi$. For y in $U - Y$, $x_1(y) \neq 0$ so

$$\begin{aligned} -\psi(y) &= \frac{\int_0^{x_1(y)} (\partial\varphi/\partial x_1)(t, x_2(y), \ldots, x_N(y))\, dt}{\int_0^{x_1(y)} (\partial l/\partial x_1)(t, x_2(y), \ldots, x_N(y))\, dt} \\ &= \varphi(y)/(l(y) - 1). \end{aligned}$$

Thus $\varphi = (1 - l)\psi$ on $U - Y$ and since all the functions in this equation are continuous, it must continue to hold on all of U, proving the result.

PROPOSITION 2. $l \in C_F(X)$.

PROOF. For any function φ and section s of L we have $\partial^F(\varphi s) = \varphi\partial^F s + d^F\varphi \otimes s$. Applying this to (8) with $\varphi = 1 - l$ we have

$$(1 - l)\partial^F s - d^F l \otimes s = \partial^F \circ J \circ \partial^F s.$$

Now applying (7) to $\partial^F s$ we have

$$(1 - l)\partial^F s = (J \circ \partial^F + \partial^F \circ J)\partial^F s = \partial^F \circ J \circ \partial^F s.$$

Thus $d^F l \otimes s = 0$. Since we can choose s to be nonzero in a neighbourhood of any point we must have $d^F l = 0$ or $l \in C_F(X)$.

COROLLARY 3. *If* $\alpha = (1 - l)\beta$ *for* α *and* β *in* $S_F^p(L)$ *then*

$$\partial^F \alpha = (1 - l)\partial^F \beta.$$

COROLLARY 4. *F is tangent to Y so $F|Y$ is a subtangent bundle on Y and all the definitions make sense with X replaced by Y.*

Define a map $\mathcal{J}$: $H^p(X;\mathbb{S}_F) \to H^{p-1}(Y;\mathbb{S}_{F|Y})$ by $\mathcal{J}[\alpha] = [J\alpha|Y]$. $\mathcal{J}$ is well defined, since restricting to Y commutes with the coboundary: for α in $S_F^p(L)$, $(\partial^F\alpha)|Y = \partial^{F|Y}\alpha|Y$, and then (7) implies $(J\partial^F\alpha)|Y + \partial^{F|Y}(J\alpha|Y) = 0$ since $l = 1$ on Y.

THEOREM 5. *$\mathcal{J}$ is an isomorphism for $p = 1$ and a monomorphism for $p \geqslant 2$.*

PROOF. First the case $p = 1$. $\mathcal{J}$ maps $H^1(X;\mathbb{S}_F)$ into $S_{F|Y}$. Suppose $\mathcal{J}[\alpha] = 0$, which means $J\alpha$ is a section of L with $\partial^F\alpha = 0$ and $J\alpha$ vanishes on Y. By Theorem 4 there is a section s of L with $J\alpha = (1 - l)s$ and so $\partial^F J\alpha = (1 - l)\partial^F s$. By (7) we then have

$$(1 - l)\partial^F s = (1 - l)\alpha - J\partial^F\alpha = (1 - l)\alpha$$

and so $\alpha = \partial^F s$ showing $[\alpha] = 0$ and $\mathcal{J}$ is injective. Now suppose s in $S_{F|Y}$. Choose any section $\tilde{s}$ of L which coincides with s on Y. This we may do since Y is a closed submanifold in X. Then $(\partial^F\tilde{s})|Y = \partial^{F|Y}\tilde{s}|Y = \partial^{F|Y}s = 0$ so $\partial^F\tilde{s}$ vanishes on Y and, hence, there is a form α in $S_F^1(L)$ with $\partial^F\tilde{s} = (1 - l)\alpha$. Clearly $\partial^F\alpha = 0$. Also $J(1 - l)\alpha = (1 - l)J\alpha$ since $l \in C_F(X)$, and so $(1 - l)J\alpha = J\partial^F\tilde{s} = (1 - l)\tilde{s}$. Thus $\tilde{s} = J\alpha$ and, hence, $\mathcal{J}[\alpha] = s$ showing $\mathcal{J}$ is onto in degree one.

For $p \geqslant 2$ suppose $\mathcal{J}[\alpha] = 0$. That is $J\alpha|Y = \partial^{F|Y}\beta$ for some β in $S_{F|Y}^{p-2}(L|Y)$. Extend β to an element $\tilde{\beta}$ in $S_F^{p-2}(L)$; then $J\alpha - \partial^F\tilde{\beta}$ vanishes on Y so there is γ in $S_F^{p-1}(L)$ with $J\alpha - \partial^F\tilde{\beta} = (1 - l)\gamma$. Then

$$\partial^F J\alpha = (1 - l)\partial^F\gamma$$

and, since $\partial^F\alpha = 0$, we have $(1 - l)\alpha = (1 - l)\partial^F\gamma$ or $\alpha = \partial^F\gamma$, showing $[\alpha] = 0$ and $\mathcal{J}$ is a monomorphism.

REMARK 4. Often in applications l has the form $\exp 2\pi ih$ for a function h on X; then Y is the union $\bigcup_{m\in\mathbf{Z}} Y_m$ with $Y_m = h^{-1}(m)$. h is in C_F and so operates by multiplication on both $H^p(X;\mathbb{S}_F)$ and $H^{p-1}(Y;\mathbb{S}_{F|Y})$ commuting with $\mathcal{J}$. Clearly

$$H^{p-1}(Y;\mathbb{S}_{F|Y}) = \prod_{m\in\mathbf{Z}} H^{p-1}(Y_m;\mathbb{S}_{F|Y_m})$$

and h has eigenvalues m with eigenspaces $H^{p-1}(Y_m;\mathbb{S}_{F|Y_m})$ when nonzero.

7. **Passing to the quotient.** Let Y be a manifold, τ: $Y \to Z$ a submersion of Y onto Z. Suppose $\tilde{F}$ is a strongly integrable subtangent bundle on Z; then $F = \tau_*^{-1}\tilde{F}$ is a strongly integrable subtangent bundle on Y. In this case we say F and $\tilde{F}$ are τ-related. Denote by V_τ all tangents ξ to Y with $\tau_*\xi = 0$. V_τ is the

bundle of vertical tangents and $V_\tau \subset F$. A vector field ξ on Y will be called vertical if ξ_x is in $V_{\tau,x}$ for all x in Y.

If F and $\tilde{F}$ are τ-related we have a map $\tau^*: \Omega^p_{\tilde{F}}(Z) \to \Omega^p_F(Y)$ for each $p \geqslant 0$ defined by

$$(\tau^*\alpha)_x(\xi_1, \ldots, \xi_p) = \alpha_{\tau x}(\tau_* \xi_1, \ldots, \tau_* \xi_p)$$

for all $\xi_1, \ldots, \xi_p$ in F_x, x in Y where $\alpha \in \Omega^p_{\tilde{F}}(Z)$. τ^* is a homomorphism of exterior algebras and intertwines d^F and $d^{\tilde{F}}$. Moreover if ξ in $\mathcal{U}_F(Y)$ is τ-related to η in $\mathcal{U}_{\tilde{F}}(Z)$, then $i(\xi) \circ \tau^* = \tau^* \circ i(\eta)$ and $\theta(\xi) \circ \tau^* = \tau^* \circ \theta(\eta)$. In particular, since a vertical vector field is τ-related to zero, $i(\xi)\tau^*\alpha = 0$, $\theta(\xi)\tau^*\alpha = 0$ for all vertical ξ, and all α in $\Omega^p_{\tilde{F}}(Z)$. In fact one may easily prove

PROPOSITION 3. *β in $\Omega^p_F(Y)$ is of the form $\beta = \tau^*\alpha$ for some α in $\Omega^p_{\tilde{F}}(Z)$ if and only if $i(\xi)\beta = 0$, $\theta(\xi)\beta = 0$ for all vertical ξ.*

We shall now extend these results to line bundle valued forms. Suppose τ: $Y \to Z$ is a submersion with connected fibres and F and $\tilde{F}$ are τ-related, strongly integrable subtangent bundles on Y and Z respectively. Let π: $L \to Y$ be a line bundle with a flat F-connection ∇; then a submanifold $M \subset Y$ is said to be absolutely parallel if $TM \subset F|M$, and parallel transport along curves in M depends only on the endpoints and not the path taken. Equivalently, the restriction of L to M with the induced connection (which is an ordinary connection since the whole tangent space TM is in F) has trivial holonomy groups at each point. We shall assume that each fibre of τ is an absolutely parallel submanifold of Y.

Define an equivalence relation $\sim$ on L by $q_1 \sim q_2$ if and only if there is a curve γ in Y with $\tau \circ \gamma$ constant and parallel transport along γ takes q_1 to q_2. Let $\tilde{L}$ denote the set of equivalence classes and $[q]$ the class containing q. Observing that $\tau(\pi(q))$ depends only on $[q]$, we define $\tilde{\tau}$: $\tilde{L} \to Z$ by $\tilde{\tau}[q] = \tau(\pi(q))$. To show $\tilde{L}$ is a line bundle over Z with a flat $\tilde{F}$-connection $\tilde{\nabla}$ induced by ∇, we need the following basic lemma.

LEMMA 1. *Let τ: $Y \to Z$ be a submersion of Y onto Z with connected fibres, F and $\tilde{F}$ τ-related strongly integrable subtangent bundles and π: $L \to Y$ a line bundle with flat F-connection ∇ such that the fibres of τ are absolutely parallel; then given z in Z there is a neighbourhood V of and a section s: $\tau^{-1}(V) \to L$ which vanishes nowhere with $\nabla s = 0$.*

PROOF. Take x_0 in Y with $\tau(x_0) = z$. By Corollary 2 there is a neighbourhood U_0 of x_0 with a section s_0: $U_0 \to L$ vanishing nowhere with $\nabla s_0 = 0$. Let $V = \tau(U_0)$. Extend s_0 to a section s on all of $\tau^{-1}(V)$ as follows. Given x in $\tau^{-1}(V)$, the connectedness of the fibres of τ implies the existence of a curve γ from a point of U_0 to x with $\tau \circ \gamma$ constant. We define $s(x)$ as the parallel transport of $s_0(\gamma(0))$ along γ. $s(x)$ is independent of the curve γ chosen since the fibres of τ are absolutely parallel. s vanishes nowhere since parallel transport is a linear isomorphism and s_0 vanishes nowhere. We must show s is smooth and $\nabla s = 0$.

Suppose γ lies wholly in an open set $U_1 \subset \tau^{-1}(V)$ with a nowhere vanishing section $s_1 : U_1 \to L$ with $\nabla s_1 = 0$. Then $U_1 \cap U_0$ contains $\gamma(0)$ so is open and nonempty. Replacing U_1 by $U_1 \cap \tau^{-1}(\tau(U_1 \cap U_0))$ if necessary, we can ensure also $U_1 \subset \tau^{-1}(\tau(U_1 \cap U_0))$. $s|U_1 = fs_1$ for some function (not yet proven to be smooth) f on U_1. Since $s|U_1 \cap U_0 = s_0|U_1 \cap U_0$, f is smooth on $U_1 \cap U_0$. Moreover, $\nabla s = \nabla s_0 = 0$ and $\nabla s_1 = 0$ on $U_1 \cap U_0$ so $d^F f = 0$ on $U_1 \cap U_0$. But then $f = \tilde{f} \circ \tau$ on $U_1 \cap U_0$ with $\tilde{f}$ a smooth function on $\tau(U_1 \cap U_0)$ with $d^{\tilde{F}} \tilde{f} = 0$. Consider $s' = \tilde{f} \circ \tau \cdot s_1$ on U_1. It satisfies $\nabla s' = 0$ and is smooth. Since $\nabla s' = 0$, the value of s' at any point is obtained from the value at some point of $U_1 \cap U_0$ by parallel transport in the fibres of τ. But

$$s'|U_1 \cap U_0 = s|U_1 \cap U_0.$$

Then by uniqueness of parallel transport we must have $s' = s|U_1$. Thus proving $s|U_1$ is smooth and $\nabla s|U_1 = 0$.

Now let x in $\tau^{-1}(V)$ be arbitrary and γ a curve from a point of U_0 to x with $\tau \circ \gamma$ constant. By the compactness of the range of γ and Corollary 2 it can be covered by finitely many open sets $U_1, \ldots, U_n$ with nowhere vanishing sections $s_i : U_i \to L$ with $\nabla s_i = 0$ and $U_i \subset \tau^{-1}(\tau(U_i \cap U_0))$. Replacing U_0 in the argument of the previous paragraph by $U_0 \cup U_1 \cup \cdots \cup U_j$, if s is proved smooth on this set with $\nabla s = 0$ there, then the same is true on $U_0 \cup U_1 \cup \cdots \cup U_{j+1}$. By induction s is smooth on U_n and $\nabla s = 0$ there. Since x was arbitrary we have shown s is smooth on $\tau^{-1}(V)$ with $\nabla s = 0$.

Given a (set theoretic) section $\tilde{s}$ of $\tilde{L}$ we define a section $\tau^* \tilde{s}$ of L by letting $(\tau^* \tilde{s})(x)$ be the unique element of L_x with $[(\tau^* \tilde{s})(x)] = \tilde{s}(\tau(x))$. It is clear that if $\nabla s = 0$ on $\tau^{-1}(V)$ with V open in Z then $s = \tau^* \tilde{s}$ for a uniquely defined section $\tilde{s}$ of $\tilde{L}$. Suppose (V, ψ) is a chart on Z; then $(\tilde{\pi}^{-1}(V), \varphi)$ defined by $\varphi(c\tilde{s}(z)) = (\psi(z), \operatorname{Re} c, \operatorname{Im} c)$ is a chart on Z and the collection of such charts defines an atlas for $\tilde{L}$ giving it the structure of a smooth line bundle. The sections $\tilde{s}$: $V \to \tilde{L}$ such that $\tau^* \tilde{s}$ is a smooth nowhere vanishing section of L on $\tau^{-1}(V)$ with $\nabla \tau^* \tilde{s} = 0$ (which exist because of Lemma 1) give a trivialisation of $\tilde{L}$. We define a flat $\tilde{F}$-connection consistently in $\tilde{L}$ by setting $\tilde{\nabla} \tilde{s} = 0$ for such a section. Then if ξ in $\mathfrak{A}_F(V)$ is τ-related to η in $\mathfrak{A}_{\tilde{F}}(Z)$ we have $\nabla_\xi \tau^* \tilde{s} = \tau^* \tilde{\nabla}_\eta \tilde{s}$ for any smooth section $\tilde{s}$ of $\tilde{L}$ and s in $\Gamma(L)$ is of the form $\tau^* \tilde{s}$ for $\tilde{s}$ in $\Gamma(\tilde{L})$ if and only if $\nabla_\xi s = 0$ for all vertical fields ξ.

Define $\tau^* : S^p_{\tilde{F}}(\tilde{L}) \to S^p_F(L)$ by

$$(\tau^* \alpha)(\xi_1, \ldots, \xi_p) = \tau^* (\alpha(\eta_1, \ldots, \eta_p))$$

for all α in $S_{\tilde{F}}^p(\tilde{L})$, $\xi_1, \ldots, \xi_p$ in $\mathfrak{U}_F(Y)$, $\eta_1, \ldots, \eta_p$ in $\mathfrak{U}_{\tilde{F}}(Z)$ with ξ_i τ-related to η_i, $i = 1, \ldots, p$. Then we have $\partial^F \circ \tau^* = \tau^* \circ \partial^{\tilde{F}}$, $i(\xi) \circ \tau^* = \tau^* \circ i(\eta)$ and $\theta(\xi) \circ \tau^* = \tau^* \circ \theta(\eta)$ for ξ in $\mathfrak{U}_F(Y)$, η in $\mathfrak{U}_{\tilde{F}}(Z)$ with ξ τ-related to η. Let $\tilde{S}_F^p(L) = \tau^* S_{\tilde{F}}^p(\tilde{L})$. Then generalising Proposition 3 we have

PROPOSITION 4. *An element α of $S_F^p(L)$ lies in $\tilde{S}_F^p(L)$ if and only if $i(\xi)\alpha = 0$, $\theta(\xi)\alpha = 0$ for all vertical vector fields ξ.*

$\tilde{S}_F^p(L)$ is stable under ∂^F, and the corresponding cohomology groups we denote by $\tilde{H}^p(Y; \mathbb{S}_F)$. τ^* defines an isomorphism of $H^p(Z; \mathbb{S}_{\tilde{F}})$ with $\tilde{H}^p(Y; \mathbb{S}_F)$ for each p and there is a natural map of $\tilde{H}^p(Y; \mathbb{S}_F)$ into $H^p(Y; \mathbb{S}_F)$ which need not be one-one nor onto.

Now we return to the situation of the previous section where σ_t is a one-parameter group on a manifold X which is periodic and generated by ξ in $\mathfrak{U}_F(X)$ for F a strongly integrable subtangent bundle. Also $\pi\colon L \to X$ is a line bundle with flat F-connection ∇ and the parallel transport function l around orbits of σ_t is T-valued with 1 a regular value. $Y = l^{-1}(1)$ consists of all orbits which are absolutely parallel submanifolds. We shall suppose the set Z of orbits of σ_t on Y is a smooth manifold with natural projection $\tau\colon Y \to Z$.

THEOREM 6. *If α is in $S_F^p(L)$ with $\partial^F\alpha = 0$ then $J\alpha|Y$ is in $\tilde{S}_{F|Y}^p(L|Y)$ and $\tau^{*-1} \circ \mathcal{J}\colon H^p(X; \mathbb{S}_F) \to H^{p-1}(Z; \mathbb{S}_{\tilde{F}})$ is an isomorphism for $p = 1$ and injective for $p > 1$.*

PROOF. $J\alpha = i(\xi)I\alpha$ and ξ spans the vertical tangent space at each point. Since $\partial^{F|Y}J\alpha|Y = 0$, it immediately follows that $J\alpha|Y$ is in $\tilde{S}_{F|Y}^p(L|Y)$ and, since $\partial^{F|Y}J\alpha|Y = 0$, it defines a class $[J\alpha|Y] = \mathcal{J}[\alpha]$ in $\tilde{H}^p(Y; \mathbb{S}_{F|Y})$. Then $\tau^{*-1} \circ \mathcal{J}[\alpha]$ is in $H^{p-1}(Z; \mathbb{S}_{\tilde{F}})$ where $\tilde{F}$ is the quotient of F by σ_t which exists since ξ is in $\mathfrak{U}_F(X)$ and F is invariant under σ_t. The remainder of the theorem follows from Theorem 5.

8. **The n-dimensional harmonic oscillator** ($n > 1$). In this example we study the quantisation of the n-dimensional harmonic oscillator using a polarisation considered by Simms [16], [17]. We shall compute the cohomology groups $H^p(X; \mathbb{S}_F)$ of this polarisation, showing that they vanish for $p \neq 1$, whilst on $H^1(X; \mathbb{S}_F)$ the Hamiltonian acts with the correct spectrum and multiplicities.

The phase space of this example is $X = \mathbf{C}^n - \{0\}$, and we let $z_1, \ldots, z_n$ be the usual linear coordinates. The symplectic form ω is $(2i)^{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j = d\alpha_0$ with $\alpha_0 = -(2i)^{-1} \sum_{j=1}^n \bar{z}_j dz_j$. As Hamiltonian we take $h = \pi \sum_{j=1}^n |z_j|^2$ with corresponding Hamiltonian vector field $\xi_h = 2\pi i \sum_{j=1}^n (z_j \partial_j - \bar{z}_j \bar{\partial}_j)$. We use the notation ∂_j for $\partial/\partial z_j$ and $\bar{\partial}_j$ for $\partial/\partial \bar{z}_j$. The flow generated by ξ_h is $\sigma_t(z_1, \ldots, z_n) = (e^{-2\pi i t} z_1, \ldots, e^{-2\pi i t} z_n)$ which has period one. α_0 and ω are invariant under σ_t, and $\alpha_0(\xi_h) = -h$.

Let $\xi_{ij} = z_i \bar{\partial}_j - z_j \bar{\partial}_i$; then at any point, precisely $n - 1$ of these vector fields are linearly independent and $\xi_{ij}(h) = 0$ for all i and j. If U_k denotes the set $\{(z_1, \ldots, z_n) \in X | z_k \neq 0\}$, then $\xi_{ij} = z_i/z_k \xi_{kj} - z_j/z_k \xi_{ki}$. Thus $\xi_{k,1}, \ldots, \widehat{\xi_{k,k}}, \ldots, \xi_{k,n}$ are linearly independent at each point of U_k. Let F_1 be the $(n-1)$-dimensional complex subtangent bundle spanned on U_k by $\xi_{k,1}, \ldots, \widehat{\xi_{k,k}}, \ldots, \xi_{k,n}$; then since $[\xi_{ij}, \xi_{r,s}] = 0$ for all i, j, r, s, F_1 is involutive. Also $[\xi_h, \xi_{rs}] = 2\pi i \xi_{rs}$ so that $F = F_1 \oplus \mathbf{C}\xi_h$ is an n-dimensional involutive complex subtangent bundle, which is in fact a strongly integrable polarisation of (X, ω). Moreover, h is in $C_F(X)$.

Let λ be the section of F^* which is 1 on ξ_h and vanishes on F_1; then $\alpha_0 | F = -h \cdot \lambda$. Consider the n-form

$$\mu = \sum_{j=1}^{n} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dh \wedge dz_{j+1} \wedge \cdots \wedge dz_n.$$

Define ζ_α^k on U_k as z_α/z_k; then $h, \zeta_1^k, \ldots, \widehat{\zeta_k^k}, \ldots, \zeta_n^k$ give C_F-coordinates on U_k and

$$\mu | U_k = z_k^n d\zeta_1^k \wedge \cdots \wedge d\zeta_{k-1}^k \wedge dh \wedge d\zeta_{k+1}^k \wedge \cdots \wedge d\zeta_n^k$$

so that μ is a global nowhere vanishing section of $K^F = \wedge^n F^0$. Further $d\mu | U_k = n dz_k/z_k \wedge \mu | U_k$ showing $\nabla \mu | U_k = n z_k^{-1} d^F z_k \otimes \mu | U_k$. But $d^F z_k(\xi_{ij}) = 0$ whilst $d^F z_k(\xi_h) = \xi_h(z_k) = 2\pi i z_k$ which implies $d^F z_k = 2\pi i z_k \lambda | U_k$ and, hence, $\nabla \mu = 2\pi i n \lambda \otimes \mu$.

Let $N_F^{1/2}$ be the bundle of $\frac{1}{2}$-forms normal to F. (See [1], [8], [15], [18].) The only properties of $N_F^{1/2}$ needed here are that $N_F^{1/2} \otimes N_F^{1/2}$ is isomorphic to K^F and $N_F^{1/2}$ has a flat F-connection $\nabla^{1/2}$ such that $\nabla^{1/2} \otimes 1 + 1 \otimes \nabla^{1/2}$ carries over into ∇ under this isomorphism. For $n > 1$,

$$H^1(X; \mathbf{Z}_2) = 0 \quad \text{and} \quad H^2(X; \mathbf{Z}) = 0$$

which allows us to assert the existence of a global nowhere vanishing section ν of $N_F^{1/2}$ with $\nu \otimes \nu$ corresponding with μ under the isomorphism. We necessarily then have $\nabla^{1/2} \nu = \frac{1}{2} 2\pi i n \lambda \otimes \nu = \pi i n \lambda \otimes \nu$.

We take L^ω as $X \times \mathbf{C}$ which has a global nowhere vanishing section s_0 given by $s_0(x) = (x, 1)$. Let α be the connection form on L^ω with $s_0^* \alpha = \alpha_0$ and let ∇^ω be the corresponding connection so that $\nabla^\omega s_0 = 2\pi i \alpha_0 \otimes s_0$.

Setting $L = L^\omega \otimes N_F^{1/2}$ and $\nabla = \nabla^\omega | F \otimes 1 + 1 \otimes \nabla^{1/2}$ and $\nu_0 = s_0 \otimes \nu$, we have $\nabla \nu_0 = (2\pi i \alpha_0 | F + \pi i n \lambda) \otimes \nu_0 = (\pi i n - 2\pi i h)\lambda \otimes \nu_0$. Parallel-transport around the integral curve of ξ_h through x is then given by $l(x) = \exp[\pi i n - 2\pi i h(x)]$. The subset Y where $l(x) = 1$ is $\{x \in X | h(x) - n/2 \in \mathbf{Z}\} = \cup_m Y_m$ where $Y_m = h^{-1}(m + n/2)$. Since h is a positive valued function we have $m > -n/2$ for nontrivial Y_m.

By our result in the previous section there is an injection of $H^p(X; \mathbb{S}_F)$ into $\prod_{m \in \mathbf{Z}} H^p(Z_m, \mathbb{S}_{\tilde{F}(m)})$ where Z_m is the quotient of Y_m by the action of σ_t and $\tilde{F}^{(m)}$ the restriction of $\tilde{F}$ to Z_m. The Z_m are the connected components of Z. Let $L^{(m)}$ be $\tilde{L}|Z_m$.

Now $Y_m = \{(z_1, \ldots, z_n) \in \mathbf{C}^n | \pi \sum_{i=1}^n |z_i|^2 = m + n/2 > 0\}$ is a $2n-1$-sphere and the action of σ_t on Y_m is the diagonal rotation of each complex variable. The map $\rho: Y_m \to Z_m$, therefore, is just the standard fibering of S^{2n-1} over $P_{n-1}(\mathbf{C})$ with fibre S^1. $\tilde{F}^{(m)}$ is the usual complex structure on $P_{n-1}(\mathbf{C})$ and $\tilde{L}|Z_m$ is a holomorphic line bundle. It must therefore be some power of the positive generator η of $H^2(P_{n-1}(\mathbf{C}), \mathbf{Z})$ say $L^{(m)} = \eta^N$. η has transition functions z_j/z_i on $U_i \cap U_j$. One easily sees this implies the pull-back $\tilde{\eta}$ of η to Y_m is trivial with a global section $\tilde{s}$ such that $\nabla \tilde{s} = -2\pi i \lambda \otimes \tilde{s}$ and, hence, η^N pulls back to $\tilde{\eta}^N$ with a section $\tilde{s}^N$ such that $\nabla \tilde{s}^N = -2\pi i N \lambda \otimes \tilde{s}^N$. But $L|Y_m$ has the global section ν_0 and $\nabla \nu_0 = (\pi i n - 2\pi i h)\lambda \otimes \nu_0 = -2\pi i m \lambda \otimes \nu_0$ on Y_m, showing $N = m$ and $L^{(m)} \cong \eta^m$.

In order to compute $H^p(Z_m; \mathbb{S}_{\tilde{F}(m)})$ we use the results on the vanishing theorem of Kodaira together with Serre duality which may be found, for instance, in [3]. Kodaira's theorem implies $H^p(Z_m, \mathbb{S}_{\tilde{F}(m)}) = 0$ for all $p < n-1$ if $m < 0$, whilst Serre duality applied to this result implies $H^p(Z_m; \mathbb{S}_{\tilde{F}(m)}) = 0$ for all $p > 0$ if $-m - n < 0$, that is, $m > -n$. Thus for $m > -n/2$ we have $H^p(Z_m; \mathbb{S}_{\tilde{F}(m)}) = 0$ for all p if $m < 0$ and $H^0(Z_m; \mathbb{S}_{\tilde{F}(m)})$ is the only nonvanishing group for $m \geqslant 0$. Its dimension is $\binom{n+m-1}{m}$. Thus since $H^p(X; \mathbb{S}_F)$ injects into $\prod_{m > -n/2} H^{p-1}(Z_m; \mathbb{S}_{\tilde{F}(m)})$, we must have $H^p(X; \mathbb{S}_F) = 0$ for $p \geqslant 2$, and for $p = 1$ we have an isomorphism $H^1(X; \mathbb{S}_F) \cong \prod_{m \geqslant 0} H^0(Z_m; \mathbb{S}_{\tilde{F}(m)})$. Now h is in $C_F(X)$, so operates on the sheaf. Under this isomorphism h operates on the right-hand side also. Its value on Y_m is $m + n/2$ and so h acts by multiplication by $m + n/2$ on the mth factor in the product. This is the correct spectrum with multiplicities for the quantum mechanical harmonic oscillator.

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